Exercise 1: Complex Number Inverse

Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$\frac{1}{a+bi} = c + di.$$

$$\frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

$$\Longleftrightarrow c + di = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

$$\Longrightarrow \begin{cases} c = \dfrac{a}{a^2+b^2} \\ d = -\dfrac{b}{a^2+b^2} \end{cases}$$

True if $a \neq 0 \wedge b \neq 0$.

Exercise 2: Cube Root of Unity

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{(-1+i\sqrt{3})^2(-1+\sqrt{3}i)}{8}$$

$$= \frac{(1-2i\sqrt{3}-3)(-1+\sqrt{3}i)}{8}$$

$$= \frac{-1+2i\sqrt{3}+3+i\sqrt{3}+2\cdot 3-3i\sqrt{3}}{8}$$

$$= \frac{8+0i}{8}$$

$$= 1$$

$\frac{-1+i\sqrt{3}}{2}$ is indeed a cube root of 1.

Exercise 3: Square Roots of i

Find two distinct square roots of i (i.e., find two different complex numbers z such that $z^2 = i$).

$$z^2 = i \iff z^2 - i = 0$$

$$\Delta = b^2 - 4ac$$

$$= 0 - 4 \cdot 1 (-1)$$

$$= 4i$$

$$z_1 = \frac{-b^2 - \sqrt{\Delta}}{2a} = \frac{-\sqrt{4i}}{2} = \frac{-\sqrt{2} - \sqrt{2}i}{2} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$z_2 = \frac{-b^2 + \sqrt{\Delta}}{2a} = \frac{\sqrt{2i}}{2} = \frac{\sqrt{2} + \sqrt{2}i}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

Exercise 4

Prove the following properties and name them:

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.
2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.
3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.
4. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.
5. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.
6. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

1. Let: $\alpha = a + bi$ and $\beta = x + yi$ where $a, b, y, x \in \mathbb{R}$

$$\alpha + \beta = (a+bi) + (x+yi) = (a+x) + (b+y)i$$

$$\beta + \alpha = (x+yi) + (a+bi) = (a+x) + (b+y)i$$

Both: $\alpha + \beta$ and $\beta + \alpha$ give the same result, therefore:

$$\alpha + \beta = \beta + \alpha \quad , \text{ when } \alpha, \beta \in \mathbb{C}$$

$\Rightarrow$ Commutativity of complex numbers

2. Let: $\alpha = a + bi$

$\beta = c + di$

$\lambda = e + fi$

where: $a, b, c, d, e, f \in \mathbb{R}$

$$(\alpha + \beta) + \lambda = (a + bi + c + di) + e + fi$$

$$= (a + c + e) + (b + d + f)i$$

moreover:

$$\alpha+(\beta+\lambda) = a+bi+(c+di+e+fi)$$

$$= (a+c+e)+(b+d+f)i$$

$\Rightarrow (\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$, where: $\alpha+\beta+\lambda \in \mathbb{C}$

$\Rightarrow$ Associativity of complex addition

3. Let: $\alpha = a+bi$
$\beta = c+di$, where $a,b,c,d,e,f \in \mathbb{R}$
$\lambda = e+fi$

$$(\alpha\beta)\lambda = [(a+bi)(c+di)](e+fi)$$

$$= (ac+adi+bci-bd)(e+fi)$$

$$= \underline{ace}+adei+bcei\underline{-bde}+acfi\underline{-adf}\underline{-bcf}-bdfi$$

$$= (ace-bde-adf-bcf)+(ade+bce+acf-bdf)i$$

$$\alpha(\beta\lambda) = (a+bi)[(c+di)(e+fi)]$$

$$= (a+bi)(ce+cfi+dei-df)$$

$$= \underline{ace}+acfi+adei\underline{-adf}+bcei\underline{-bcf}\underline{-bde}-bdfi$$

$= (ace - adf - bcf - bde) + (acf + ade + bce - bdf)i$

From this we observe that: $(\alpha B)\lambda = \alpha(B\lambda)$ where $\alpha, B, \lambda \in \mathbb{C}$

$\Rightarrow$ Associativity of complex multiplication

4. Let $\alpha = a + bi$ and $B = -a - bi$ where $a, b, c, d \in \mathbb{R}$

$$\alpha + B = (a+bi) + (-a-bi)$$

$$= a - a + bi - bi$$

$$= 0$$

This means there exists at least one additive inverse for α.

Now suppose $\exists B' \in \mathbb{C} : B = B'$, we get:

$$\begin{cases} \alpha + B = 0 & ① \\ \alpha + B' = 0 & ② \end{cases}$$

$① - ②$: $\alpha + B - (\alpha + B') = 0 \iff \alpha - \alpha + B - B' = 0$

$$\iff B = B'$$

In conclusion: $\forall \alpha, \exists B : \alpha + B = 0$

Also: B is a unique additive inverse.

5. Given $\alpha \in \mathbb{C}^*$ where $\alpha = a+bi$ $a, b \in \mathbb{R}$ suppose $B \in \mathbb{C}$ where $B = \frac{1}{a+bi}$:

$$\alpha B = (a+bi)\frac{1}{a+bi}$$

$$= (a+bi)\left(\frac{1}{a+bi} \cdot \frac{a-bi}{a-bi}\right)$$

$$= (a+bi)\left(\frac{a-bi}{a^2+b^2}\right)$$

$$= \frac{a^2-abi+abi+b^2}{a^2+b^2}$$

$$= \frac{a^2+b^2}{a^2+b^2}$$

$$= 1$$

There then exists at least one <u>multiplicative inverse</u>.

Suppose now $\exists\, \beta'$ such that $\alpha\beta' = 1$

take: $\alpha\beta = 1 \quad \Big| \cdot \beta'$

$$\Leftrightarrow \alpha \cdot \beta \cdot \beta' = \beta'$$

$$\Leftrightarrow (\alpha \cdot \beta') \cdot \beta = \beta'$$

$$\Leftrightarrow 1 \cdot \beta = \beta'$$

$$\Leftrightarrow \beta = \beta'$$

The solution for the <u>multipliative inverse</u> therefore is <u>unique</u>.

6. Let $\alpha, \beta, \lambda \in \mathbb{C}$ where: $\alpha = a+bi \quad a,b,c,d,e,f \in \mathbb{R}$

$\beta = c+di$

$\lambda = e+fi$

$$\lambda(\alpha+\beta) = (e+fi)\left[(a+bi)+(c+di)\right]$$

$$= (e+fi)(a+c+bi+di)$$

$$= ae + ce + bei + dei + afi + cfi - bf - df$$

$$= (ae + ce - bf - df) + (be + de + af + cf)i$$

Comparing this to:

$$\lambda\alpha + \lambda\beta = (e+fi)(a+bi) + (e+fi)(c+di)$$

$$= ae + bei + afi - bf + ce + dei + cfi - df$$

$$= (ae - bf + ce - df) + (be + af + de + cf)i$$

Both of these expression give the same complex number, they must therefore be equal:

$$\underline{\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta}$$

Exercise 5: Vector Equation in $\mathbb{R}^4$

Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

We have:

$$\begin{pmatrix} 4 \\ -3 \\ 1 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ -6 \\ 8 \end{pmatrix} \Rightarrow \begin{cases} 4 + 2x_1 = 5 & (1) \\ -3 + 2x_2 = 9 & (2) \\ 1 + 2x_3 = -6 & (3) \\ 7 + 2x_4 = 8 & (4) \end{cases}$$

(1): $4 + 2x_1 = 5 \Leftrightarrow 2x_1 = 5 - 4$

$\Leftrightarrow x_1 = \frac{1}{2}$

(2): $-3 + 2x_2 = 9 \Leftrightarrow 2x_2 = 12$

$\Leftrightarrow x_2 = 6$

(3): $1 + 2x_3 = -6 \Leftrightarrow x_3 = -\frac{7}{2}$

(4): $7 + 2x_4 = 8 \Leftrightarrow x_4 = \frac{1}{2}$

We get x to be: $x = \begin{pmatrix} \frac{1}{2} \\ 6 \\ -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix}$

Exercise 6: Non-Existence of Scalar Multiplication

Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

We have:

$$\lambda \begin{pmatrix} 2-3i \\ 5+4i \\ -6+7i \end{pmatrix} = \begin{pmatrix} 12-5i \\ 7+22i \\ -32-9i \end{pmatrix}$$

$\lambda(2-3i) = 12-5i \Leftrightarrow \lambda = \frac{12-5i}{2-3i}$

$= \frac{(12-5i)(2+3i)}{2^2+3^2}$

$= \frac{24 + 36i - 10i + 15}{13}$

$$= \frac{39+28i}{13}$$

$$= 3+2i$$

If λ <u>for each</u> of the equations does not equal this result the assumption in the exercise is true. <u>λ can not be multi valued.</u>

$$\lambda(5+4i) = 7+22i \iff \lambda = \frac{7+22i}{5+4i}$$

$$= \frac{(7+22i)(5-4i)}{5^2+4^2}$$

$$= \frac{35-28i+110i+88}{41}$$

$$= \frac{123+82i}{41}$$

$$= 3+2i$$

Still true.

$$\lambda(-6+7i) = -32-9i \iff \lambda = -\frac{32+9i}{-6+7i}$$

$$= -\frac{(32+9i)(-6-7i)}{6^2+7^2}$$

$$= -\frac{-192-224i-54i+63}{85}$$

$$= -\frac{-129-278i}{85}$$

$$= \frac{129+278i}{85}$$

Not all values for λ are equal, there does not exist a unique λ for which, the assumption is true.

Exercise 7

Prove the following:

1. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}^n$.
2. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.
3. Show that $1x = x$ for all $x \in \mathbb{F}^n$, where 1 is the multiplicative identity in $\mathbb{F}$.
4. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.
5. Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

1. Let: $x = (x_1, x_2, x_3, \dots, x_n)$

$y = (y_1, y_2, \dots, y_n)$

$z = (z_1, z_2, \dots, z_n)$

$$(x+y)+z = [(x_1+y_1)+z_1, \dots, (x_n+y_n)+z_n]$$

Since addition of scalars is associative this is equivalent to:

$$= [x_1+(y_1+z_1), \dots, x_n+(y_n+z_n)]$$

$$= x+(y+z)$$

$\square$

2. Let $x = (x_1, x_2, \dots, x_n)$ and $a, b \in \mathbb{F}$

$$(ab)x = (ab)(x_1, x_2, \dots, x_n)$$

Since multiplication of scalars is commutative, this is equivalent to:

$$= a(bx_1, bx_2, \dots, bx_n)$$

$$= a(bx)$$

$\square$

3. Let: $x = (x_1, x_2, \dots, x_n)$

$$1 \cdot x = 1 \cdot (x_1, \dots, x_n)$$

$$= (1 \cdot x_1, \dots, 1 \cdot x_n)$$

$$= (x_1, \dots, x_n)$$

$$= x$$

$\square$

4. Let: $\lambda \in \mathbb{F}$ and $x = (x_1, x_2, \dots, x_n)$, $y = (x_1, x_2, \dots, x_n)$

$$\lambda(x+y) = \lambda((x_1, \dots, x_n) + (y_1, \dots, y_n))$$

$$= \lambda(x_1, \dots x_n) + \lambda(y_1, \dots, y_n)$$

$$= \lambda \cdot x + \lambda \cdot y$$

$\square$

5. Let: $x = (x_1, \dots, x_n)$ and $a, b \in \mathbb{F}$

$$
\begin{aligned}
(a+b)x &= (a+b)(x_1, \dots, x_n) \\
&= \big((a+b)x_1, \dots, (a+b)x_n\big) \\
&= (ax_1 + bx_1, \dots, ax_n + bx_n) \\
&= (ax_1, \dots, ax_n) + (bx_1, \dots, bx_n) \\
&= ax + bx
\end{aligned}
$$

$\square$