Exercises

Luxformel

Definition of Vector Space

Exercise 1: Double Additive Inverse

Prove that $-(-v) = v$ for every $v \in V$, where V is a vector space.

Exercise 2: Zero Product Property

Suppose $a \in \mathbb{F}$ (a field), $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Exercise 3: Unique Solution to Vector Equation

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Exercise 4: Alternative Axiom for Additive Inverses

Show that in the definition of a vector space, the additive inverse condition can be replaced with:

$$0v = 0 \text{ for all } v \in V$$

(where left $0 \in \mathbb{F}$, right $0 \in V$).

Exercise 5: Extended Real Numbers as a Vector Space?

Define **addition** and **scalar multiplication** on $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ as follows:

For $t \in \mathbb{R}$:

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \text{(analogous)}$$

For $t \in \mathbb{R}$:

$$t + \infty = \infty + t = \infty, \quad \infty + \infty = \infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbb{R}^*$ a vector space over $\mathbb{R}$? Justify.