Exercise 1

Consider the following vectors in $\mathbb{R}^7$:

$$u = (0.5, 0.4, 0.4, 0.5, 0.1, 0.4, 0.1), \quad v = (-1, -2, 1, -2, 3, 1, -5)$$

1. Check if u and v are unit vectors.

2. Calculate the dot product of the vectors u and v.

3. Are u and v orthogonal?

1. u is a unit vector if $\|\vec{u}\| = 1$

$$\|\vec{u}\| = \sqrt{0{,}5^2 + 0{,}4^2 + 0{,}4^2 + 0{,}5^2 + 0{,}1^2 + 0{,}4^2 + 0{,}1^2}$$

$$= 1$$

u is a unit vector.

v is a unit vector if $\|\vec{v}\| = 1$

$$\|\vec{v}\| = \sqrt{(-1)^2 + (-2)^2 + 1^2 + (-2)^2 + 3^2 + 1^2 + (-5)^2}$$

$$= 3\sqrt{5}$$

v is not a unit vector.

2. $$\vec{u} \circ \vec{v} = 0{,}5 \cdot (-1) + 0{,}4 \cdot (-2) + 0{,}4 \cdot 1 + 0{,}5 \cdot (-2) + 0{,}1 \cdot 3 + 0{,}4 \cdot 1 + 0{,}1 \cdot 5$$

$$= -1{,}7$$

3. Since $\vec{u} \odot \vec{v}$ is not 0, $\vec{u}$ and $\vec{v}$ are not orthogonal

Exercise 2

Consider the following vectors in $\mathbb{R}^9$:

$$u = (1, 2, 5, 2, -3, 1, 2, 6, 2),$$

$$v = (-4, 3, -2, 2, 1, -3, 4, 1, -2)$$

$$w = (3, 3, -3, -1, 6, -1, 2, -5, -7)$$

1. Which pairs of these vectors are orthogonal?

2. Calculate the Euclidean norm of u.

3. Calculate the infinity norm of w.

1. All possible combinations: u, v ; u, w ; v, w

$$\vec{u} \odot \vec{v} = 1 \cdot (-4) + 2 \cdot 3 + 5 \cdot (-2) + 2 \cdot 2 - 3 \cdot 1 + 1 \cdot (-3) + 2 \cdot 4 + 6 \cdot 1 + 2 \cdot (-2)$$

$$= 0$$

$\vec{u}$ and $\vec{v}$ are orthogonal.

$$\vec{u} \odot \vec{w} = 1 \cdot 3 + 2 \cdot 3 + 3 \cdot (-5) + 2 \cdot (-1) - 3 \cdot 6 + 1 \cdot (-1) + 2 \cdot 2 + 6 (-5) + 2 \cdot (-7)$$

$$= -67$$

$\vec{u}$ and $\vec{w}$ are not orthogonal.

$$\vec{v} \odot \vec{w} = -4 \cdot 3 + 3 \cdot 3 - 2 \cdot (-3) + 2 \cdot (-1) + 1 \cdot 6 - 3 \cdot (-1) + 4 \cdot 2 + 1 \cdot (-5) - 2 \cdot (-7)$$

$= 27$

$\vec{v}$ and $\vec{w}$ are not orthogonal.

2. $\|\vec{u}\| = \sqrt{1^2 + 2^2 + 5^2 + 2^2 + 3^2 + 1^2 + 2^2 + 6^2 + 2^2}$

$= 2\sqrt{22}$

3. $\|\vec{w}\|_\infty = \max(|3|, |3|, |-3|, |-1|, |6|, |-1|, |2|, |-5|, |-7|)$

$= 7$

Exercise 3

Consider the following matrices:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad F = \begin{pmatrix} -2 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix}$$

1. Calculate, if possible:
   - $A + B$
   - $B - A$
   - $B + C$
   - AB
   - BA
   - BG
   - CE
   - EF
   - FE
2. Write the transposes of A and B and calculate their product. Which property can one observe?

1. $A+B = \begin{pmatrix} 2+3 & -2+1 \\ 0+6 & 1+2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 6 & 3 \end{pmatrix}$

$B - A = \begin{pmatrix} 3-2 & 1-(-2) \\ 6-0 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 6 & 1 \end{pmatrix}$

$B+C$ not possible!

$$A \cdot B = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 3 - 2\cdot 6 & 2\cdot 1 - 2\cdot 2 \\ 3\cdot 0 + 1\cdot 6 & 0\cdot 1 + 1\cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -2 \\ 6 & 2 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 2 + 1\cdot 0 & -3\cdot 2 + 1\cdot 1 \\ 6\cdot 2 + 2\cdot 0 & -6\cdot 2 + 2\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 \\ 12 & -10 \end{pmatrix}$$

$$B \cdot G = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 1+1\cdot 1 & -3\cdot 1+1\cdot 4 & 0 & 0 \\ 6\cdot 1+2\cdot 1 & -6\cdot 1+2\cdot 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 0 & 0 \\ 8 & 2 & 0 & 0 \end{pmatrix}$$

$$C\cdot E = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4\cdot 2 - 1\cdot 1 - 1\cdot 3 \\ 2\cdot 2 - 5\cdot 1 - 2\cdot 3 \\ 1\cdot 2 - 1\cdot 1 + 2\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -7 \\ 7 \end{pmatrix}$$

$$E\cdot F = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ -6 & 3 & 0 \end{pmatrix}$$

$$F \cdot E = \begin{pmatrix} -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \cdot 2 - 1 \cdot 1 + 0 \cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \end{pmatrix}$$

2. $A^T = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$; $B^T = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$

$$A^T \cdot B^T = \begin{pmatrix} 2 \cdot 3 + 0 & 2 \cdot 6 + 0 \\ -2 \cdot 3 + 1 \cdot 1 & -2 \cdot 6 + 2 \cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 \\ -5 & -10 \end{pmatrix}$$

$$= (A \cdot B)^T$$

Observation: $A^T \cdot B^T = (A \cdot B)^T$

Exercise 4

Consider the following matrices:

$$A = \begin{pmatrix} 2 & -2 \\ -3 & 1 \\ 5 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 & 4 \\ -2 & 3 & -7 \\ 2 & 5 & -7 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -1 & 2 \\ -8 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix}$$

1. Compute $A^T B$ and $C + B$.

2. Which of the matrices A, B, C are full rank?

3. Calculate the Frobenius norm of C and the spectral norm of A.

4. Calculate the inverse of B.

1. $A^T \cdot B = \begin{pmatrix} 2 & -3 & 5 \\ -2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 4 & 4 & 4 \\ -2 & 3 & -7 \\ 2 & 5 & -7 \end{pmatrix}$

$= \begin{pmatrix} 2\cdot 4 + 3\cdot 2 + 5\cdot 2 & 2\cdot 4 - 3\cdot 3 + 5\cdot 5 & 2\cdot 4 + 3\cdot 7 - 5\cdot 7 \\ -2\cdot 4 - 2\cdot 1 - 3\cdot 2 & -2\cdot 4 + 1\cdot 3 - 3\cdot 5 & -2\cdot 4 - 7\cdot 1 + 3\cdot 7 \end{pmatrix}$

$= \begin{pmatrix} 24 & 24 & -6 \\ -16 & 20 & 6 \end{pmatrix}$

$C + B = \begin{pmatrix} 4 & 1 & 2 \\ -8 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} 4 & 4 & 4 \\ -2 & 3 & -7 \\ 2 & 5 & -7 \end{pmatrix}$

$= \begin{pmatrix} 8 & 5 & 6 \\ -10 & 5 & -11 \\ 4 & 6 & -11 \end{pmatrix}$

2. Full rank means all columns are linear independent.

A has two linearly independent columns so: $\text{rank}(A) = 2$

A has full rank.

rank (B) = 3 , full rank

for C: $C = \begin{pmatrix} — r_1 — \\ — r_2 — \\ — r_3 — \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ -8 & 2 & -4 \\ 2 & 1 & -4 \end{pmatrix}$

notice: $r_2 = 2 \cdot r_1$

C is not full rank

3. Frobenius norm:

$$\|C\|_F = \sqrt{\sum_{ij} c_{ij}^2}$$

$$= \sqrt{4^2 + 1^2 + 2^2 + 8^2 + 2^2 + 4^2 + 2^2 + 1^2 + 4^2}$$

$$= 3\sqrt{14}$$

Spectral norm:

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$$

$$A^T \cdot A = \begin{pmatrix} 2 & -3 & 5 \\ -2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -3 & 1 \\ 5 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 2 + 9 + 5 \cdot 5 & -2 \cdot 2 - 3 \cdot 1 - 5 \cdot 3 \\ -2 \cdot 2 - 3 \cdot 1 - 3 \cdot 5 & 2 \cdot 2 + 1 \cdot 1 + 3 \cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & -22 \\ -22 & 14 \end{pmatrix}$$

<u>Eigenvalues of $A^T \cdot A$:</u>

$$\det \begin{pmatrix} 38-\lambda & -22 \\ -22 & 14-\lambda \end{pmatrix} = 0$$

$$\Leftrightarrow (38-\lambda)(14-\lambda) - 484 = 0$$

$$\Leftrightarrow 532 - 38\lambda - 14\lambda + \lambda^2 - 484 = 0$$

$$\Leftrightarrow \lambda^2 - 52\lambda + 48 = 0$$

$$\Leftrightarrow \lambda = 26 \pm 2\sqrt{157}$$

for $\forall \lambda \in \mathbb{R}$

$$\Rightarrow \lambda = 26 + 2\sqrt{157}$$

$$\|A\|_2 = \sqrt{26 + 2\sqrt{157}}$$

4. $$\left(\begin{array}{ccc|ccc} 4 & 4 & 4 & 1 & 0 & 0 \\ -2 & 3 & -7 & 0 & 1 & 0 \\ 2 & 5 & -7 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 5 & -5 & \frac{1}{2} & 1 & 0 \\ 0 & 3 & -9 & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{10} & \frac{1}{5} & 0 \\ 0 & 0 & -6 & -\frac{4}{5} & -\frac{3}{5} & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{10} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{2}{15} & \frac{1}{10} & -\frac{1}{6} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{60} & -\frac{2}{5} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{7}{36} & \frac{3}{10} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{2}{15} & \frac{1}{10} & -\frac{1}{6} \end{array}\right)$$

So $B^{-1} = \begin{pmatrix} -\frac{7}{60} & -\frac{2}{5} & \frac{1}{3} \\ \frac{7}{60} & \frac{3}{10} & \frac{1}{6} \\ \frac{2}{15} & \frac{1}{10} & -\frac{1}{6} \end{pmatrix}$

Exercise 5

Consider the following matrices:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

1. Calculate the determinants of the matrices A, B, and AB.
2. Calculate the determinants of the matrices C and D.

1. $\det(A) = 2 \cdot 1 - (-2) \cdot 0 = 2$

$\det(B) = 3 \cdot 2 - 1 \cdot 6 = 0$

$$A \cdot B = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 3 - 2\cdot 6 & 2\cdot 1 - 2\cdot 2 \\ 0\cdot 3 + 1\cdot 6 & 0\cdot 1 + 2\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -2 \\ 6 & 2 \end{pmatrix}$$

$$\det(A\cdot B) = -6\cdot 2 - (-2)\cdot 6$$

$$= 0$$

2.

$$C = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}\begin{matrix} 4 & 1 \\ 2 & 5 \\ 1 & 1 \end{matrix}$$

$$\det(C) = 4\cdot 5\cdot 2 + 1\cdot(-2)\cdot 1 + (-1)\cdot 2\cdot 1 - \left((-1)\cdot 5\cdot 1 + 4\cdot(-2)\cdot 1 + 1\cdot 2\cdot 2\right)$$

$$= 40 - 2 - 2 - (-5 - 8 + 4)$$

$$= 45$$

$$D = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

$$\det(D) = -3 \cdot \begin{vmatrix} 5 & -1 \\ 6 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -7 & -1 \\ -6 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -7 & 5 \\ -6 & 6 \end{vmatrix}$$

$$= -3\left(5 \cdot (-2) + 1 \cdot 6\right) - \left(-7 \cdot (-2) - (-1) \cdot (-6)\right) - \left(-7 \cdot 6 - (-6) \cdot 5\right)$$

$$= 12 - 8 + 12$$

$$= 16$$

Exercise 6

Consider the following matrices:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & -9 \\ -4 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 6 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & -5 \end{pmatrix}$$

Calculate, if possible, the inverses of the matrices A, B, C, and D.

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$

$$= \frac{1}{2 \cdot 3 - (-1) \cdot 4} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} 5 & 0 \\ -4 & 2 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 5 & 0 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$\det(C) = 6 \cdot 6 - 4 \cdot 9 = 0$$

C is not invertible there is <u>no inverse.</u>

$$D = \begin{pmatrix} -1 & 6 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & -5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & -5 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & -6 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 18 & 1 & 3 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & -6 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & -18 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -30 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & -18 & 1 \end{array}\right)$$

$$D^{-1} = \begin{pmatrix} 5 & -30 & 2 \\ 0 & 1 & 0 \\ 3 & -18 & 1 \end{pmatrix}$$

Exercise 7

Consider the matrix:

$$A = \begin{pmatrix} 2 & 2 & 3 \\ -2 & 7 & 4 \\ -3 & -3 & -4 \\ -8 & 2 & 3 \end{pmatrix}$$

1. Add a column to A so that it is invertible.
2. Remove a row from A so that it is invertible.
3. Calculate AA^T. Is it invertible?
4. Calculate A^TA. Is it invertible?

1. A is invertible if $\det(A) \neq 0$. Therefore:

Suppose:
$$\left(\begin{array}{cccc|ccc} 2 & 2 & 3 & a & 2 & 2 & 3 \\ -2 & 7 & 4 & b & -2 & 7 & 4 \\ -3 & -3 & -4 & c & -3 & -3 & -4 \\ -8 & 2 & 3 & d & -8 & 2 & 3 \end{array}\right)$$

$$\det(A) = 2\cdot 7\cdot(-4)\cdot d + 2\cdot 4\cdot c\cdot(-8) + 3\cdot b\cdot(-3)\cdot 2 + a\cdot(-2)\cdot(-3)\cdot 3$$
$$- \left[a\cdot 4\cdot(-3)\cdot(-8) + 2\cdot b\cdot(-4)\cdot 2 + 2\cdot(-2)\cdot c\cdot 3 + 3\cdot 7\cdot(-3)\cdot d \right]$$

$$= -56d - 64c - 18\cdot b + 18a - (96a - 16b - 12c - 63d)$$

$$= -78a - 2b - 52c + 7d$$

for A to be invertible:

$$\det(A) \neq 0 \iff -78a - 2b - 52c + 7d \neq 0$$

$$\implies -78 - 2 - 52 + 7 \neq 0$$

$a, b, c, d = 1$, works

$$A = \begin{pmatrix} 2 & 2 & 3 & 1 \\ -2 & 7 & 4 & 1 \\ -3 & -3 & -4 & 1 \\ -8 & 2 & 3 & 1 \end{pmatrix}$$

2. If we remove the last row the matrix becomes

invertible.

$$A' = \left(\begin{array}{ccc|cc} 2 & 2 & 3 & 2 & 2 \\ -2 & 7 & 4 & -2 & 7 \\ -3 & -3 & -4 & -3 & -3 \end{array}\right)$$

$$\det(A') = 2\cdot 7\cdot(-4) + 2\cdot 4\cdot(-3) + 3\cdot 2\cdot 3 - \left(3\cdot 7\cdot(-3) + 2\cdot 4\cdot(-3) + 2\cdot(-2)\cdot(-4)\right)$$

$$= -56 - 24 + 18 - \left(-63 - 24 + 16\right)$$

$$= 9 \quad (\neq 0)$$

3.

$$AA^T = \begin{pmatrix} 2 & 2 & 3 \\ -2 & 7 & 4 \\ -3 & -3 & -4 \\ -8 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -2 & -3 & -8 \\ 2 & 7 & -3 & 2 \\ 3 & 4 & -4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 2 + 2\cdot 2 + 3\cdot 3 & -2\cdot 2 + 2\cdot 7 + 3\cdot 4 & -3\cdot 2 - 3\cdot 2 - 4\cdot 3 & -2\cdot 8 + 2\cdot 2 + 3\cdot 3 \\ -2\cdot 2 - 2\cdot 7 + 4\cdot 3 & 2\cdot 2 + 7\cdot 7 + 4\cdot 4 & 2\cdot 3 - 7\cdot 3 - 4\cdot 4 & 2\cdot 8 + 2\cdot 7 + 4\cdot 3 \\ -3\cdot 2 - 3\cdot 2 - 4\cdot 3 & 3\cdot 2 - 3\cdot 7 - 4\cdot 4 & 3\cdot 3 + 8\cdot 3 + 4\cdot 4 & 3\cdot 8 - 3\cdot 2 - 4\cdot 3 \\ -8\cdot 2 + 2\cdot 2 + 3\cdot 3 & 8\cdot 2 + 2\cdot 7 + 3\cdot 4 & 3\cdot 8 - 3\cdot 2 - 3\cdot 4 & 8\cdot 8 + 2\cdot 2 + 3\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 22 & -12 & -3 \\ -6 & 69 & -31 & 42 \\ -24 & -31 & 34 & 6 \\ -3 & 42 & 6 & 77 \end{pmatrix}$$

Calculator: $\det(AA^T) = 1'103'024 \neq 0$

AA^T is invertible.

4.
$$A^TA = \begin{pmatrix} 2 & -2 & -3 & -8 \\ 2 & 7 & -3 & 2 \\ 3 & 4 & -4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ -2 & 7 & 4 \\ -3 & -3 & -4 \\ -8 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot2+2\cdot2+3\cdot3+8\cdot8 & 2\cdot2-2\cdot7+3\cdot3-8\cdot2 & 2\cdot3-2\cdot4+3\cdot4-8\cdot3 \\ 2\cdot2-2\cdot7+3\cdot3-2\cdot8 & 2\cdot2+7\cdot7+3\cdot3+2\cdot2 & 3\cdot2+7\cdot4+4\cdot3+2\cdot3 \\ 3\cdot2-2\cdot4+4\cdot3-8\cdot3 & 2\cdot3+4\cdot7+4\cdot3+2\cdot3 & 3\cdot3+4\cdot4+4\cdot4+3\cdot3 \end{pmatrix}$$

$$= \begin{pmatrix} 81 & -17 & -14 \\ -17 & 66 & 52 \\ -14 & 52 & 50 \end{pmatrix}$$

Calculator: $\det(A^TA) = 45'642 \neq 0$

A^TA is invertible.

Exercise 8

1. Calculate the inverse of the matrix $M = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -4 & 5 \end{pmatrix}$.

2. Use this inverse to solve the linear system:

$$\begin{cases} 3x + 2y - z = 5 \\ x - y + z = 1 \\ 2x - 4y + 5z = -3 \end{cases}$$

$x = 2$
$y = -2$
$z = -3$

1.
$$M = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -4 & 5 \end{pmatrix}$$

Matrix of minors:

$$\begin{pmatrix} \begin{vmatrix} -1 & 1 \\ -4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ -4 & 5 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 3 & -2 \\ 6 & 17 & -16 \\ 1 & 4 & -5 \end{pmatrix}$$

Cofactor matrix:

$$\begin{pmatrix} -1 & -3 & -2 \\ -6 & 17 & 16 \\ 1 & -4 & -5 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det(M)} \, adj(M) = \frac{1}{\det(M)} \begin{pmatrix} -1 & -3 & -2 \\ -6 & 17 & 16 \\ 1 & -4 & -5 \end{pmatrix}^T$$

<u>$\det(M)$?</u>

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -4 & 5 \end{pmatrix} \begin{matrix} 3 & 2 \\ 1 & -1 \\ 2 & -4 \end{matrix}$$

$$\det(M) = 3 \cdot (-1) \cdot 5 + 2 \cdot 1 \cdot 2 + (-1) \cdot 1 \cdot (-4) - \left(2 \cdot (-1) \cdot (-1) + (-4) \cdot 1 \cdot 3 + 5 \cdot 1 \cdot 2\right)$$

$$= -7$$

$$M^{-1} = \frac{1}{-7} \begin{pmatrix} -1 & -6 & 1 \\ -3 & 17 & -4 \\ -2 & 16 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{6}{7} & -\frac{1}{7} \\ \frac{3}{7} & -\frac{17}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{16}{7} & \frac{5}{7} \end{pmatrix}$$

2. $\begin{cases} 3x + 2y - z = 5 \\ x - y + z = 1 \\ 2x - 4y + 5z = -3 \end{cases} \Longleftrightarrow \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} \quad | \cdot M^{-1}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{6}{7} & -\frac{1}{7} \\ \frac{3}{7} & -\frac{17}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{16}{7} & \frac{5}{7} \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{7} + \frac{6}{7} + \frac{3}{7} \\ \frac{15}{7} - \frac{17}{7} - \frac{12}{7} \\ \frac{10}{7} - \frac{16}{7} - \frac{15}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$$

Exercise 9

Solve the systems:

1.

$$\begin{cases} 2x + 3y + 5z = 2 \\ 7x + z = -1 \\ -2y + 2z = 3 \end{cases}$$

2.

$$\begin{cases} x + 2y - z = 2 \\ 2x + 5y + 4z = 3 \\ 3x + 7y + 4z = 1 \end{cases}$$

1. $$\begin{cases} 2x + 3y + 5z = 2 \\ 7x \qquad\quad + z = -1 \\ \qquad -2y + 2z = 3 \end{cases}$$

Augmented matrix:

$$R_2 - \tfrac{7}{2}\cdot R_1$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 2 \\ 7 & 0 & 1 & -1 \\ 0 & -2 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 2 \\ 0 & -\frac{21}{2} & -\frac{33}{2} & -8 \\ 0 & -2 & 2 & 3 \end{array}\right)\begin{array}{l} \cdot\frac{1}{2} \\ \cdot\left(-\frac{2}{21}\right) \\ \cdot\left(\frac{1}{2}\right) \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 1 \\ 0 & 1 & \frac{11}{7} & \frac{16}{21} \\ 0 & -1 & 1 & \frac{3}{2} \end{array}\right) \quad R_3 + R_2$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 1 \\ 0 & 1 & \frac{11}{7} & \frac{16}{21} \\ 0 & 0 & \frac{18}{7} & \frac{95}{42} \end{array}\right)\begin{array}{l} \\ \\ \cdot\frac{7}{18} \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 1 \\ 0 & 1 & \frac{11}{7} & \frac{16}{21} \\ 0 & 0 & 1 & \frac{95}{108} \end{array}\right) \quad R_2 - \frac{11}{7}\cdot R_3$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{5}{2} & 1 \\ 0 & 1 & 0 & -\frac{67}{108} \\ 0 & 0 & 1 & \frac{95}{108} \end{array}\right) \quad R_1 - \frac{3}{2}R_2 - \frac{5}{2}R_3$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{29}{108} \\ 0 & 1 & 0 & -\frac{67}{108} \\ 0 & 0 & 1 & \frac{95}{108} \end{array}\right)$$

$$\Rightarrow \quad x = -\frac{29}{108}, \quad y = -\frac{67}{108}, \quad z = \frac{95}{108}$$

2. $$\begin{cases} x + 2y - z = 2 \\ 2x + 5y + 4z = 3 \\ 3x + 7y + 4z = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 5 & 4 & 3 \\ 3 & 7 & 4 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 6 & -1 \\ 0 & 1 & 7 & -5 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right) \quad R_3 - R_2$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & -4 \end{array}\right) \quad R_2 - 6R_3$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & -48 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & -4 \end{array}\right) \quad R_1 - 2R_2 + R_3$$

$\Rightarrow x = -48, \; y = 23, \; z = -4$