Exercise 1: Double Additive Inverse

Prove that $-(-v) = v$ for every $v \in V$, where V is a vector space.

Let V be some vector space over some field. By the vector space axioms, we have:

1. For every $v \in V$ there exists an additive inverse $-v$ such that:

$$v + (-v) = 0$$

Where $-v$ belongs to V; closure under addition.

2. Additive inverses are unique

Fix $v \in V$ such that $v + (-v) = 0$. By commutativity of addition, we get:

$$(-v) + v = 0$$

Now v is the additive inverse of $(-v)$. By uniqueness of additive inverses, the additive inverse of $(-v)$ is $-(-v)$, meaning:

$$-(-v) = v.$$

$\square$

Exercise 2: Zero Product Property

Suppose $a \in \mathbb{F}$ (a field), $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Let $a \in \mathbb{F}$ and $v \in V$:

If $a = 0$, then $a \cdot v = 0$, trivial case.

If $a \neq 0$, then there exists $a^{-1} \neq 0$, then.

$$a^{-1}(a \cdot v) = 0 \iff a^{-1} \cdot 0 = 0$$

But $a^{-1} \neq 0$, therefore $v = 0$.

□

Exercise 3: Unique Solution to Vector Equation

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

We have: $v + 3x = w$

$$\iff v - w = 3x$$

Since on the left hand side we only have vector addition (substraction) this vector exists in V.

Further more on the right hand side we only have scalar multiplication which also creates a valid vector in V.

Thus there must be some x which satisfies this condition.

□

Exercise 4: Alternative Axiom for Additive Inverses

Show that in the definition of a vector space, the additive inverse condition can be replaced with:

$$0v = 0 \text{ for all } v \in V$$

(where left 0 $\mathbb{F}$, right 0 V).

In any scalar field $\mathbb{F}$ we have:

$$1+(-1)=0$$

with vector multiplication:

$$v(1+(-1))=0\cdot v$$

by distributivity:

$$1v+(-1)v=0v$$

By assumption $0\cdot v=0$, thus:

$$v+(-1)\cdot v=0$$

Which is the additive inverse property, thus with the assumption it is possible to replace the additive inverse.

Exercise 5: Extended Real Numbers as a Vector Space?

Define **addition** and **scalar multiplication** on $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ as follows:

For $t \in \mathbb{R}$:

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \text{(analogous)}$$

For $t \in \mathbb{R}$:

$$t + \infty = \infty + t = \infty, \quad \infty + \infty = \infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbb{R}^*$ a vector space over $\mathbb{R}$? Justify.

Not a vector space because the distributivity property fails.

Counterexample: let $t, s \in \mathbb{R}$ and $v \in \mathbb{R}^*$

where: $t = -1$ and $s = 2$ and $v = +\infty$

Distributivity: $(s+t)\cdot v = (2+(-1))\cdot v = 1\cdot(+\infty)$

but: $(s+t)\cdot v = s\cdot v + t\cdot v = 2\cdot(+\infty) + (-1)(+\infty) = +\infty - \infty = 0$

($\neq$)

Thus $\mathbb{R}^*$ is not a vector space over $\mathbb{R}$.